

Algorithms and Execution Traces

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- 1 The big picture
- 2 Challenges
- 3 Main contributions

Algorithm The Euclidean algorithm

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2   |  $z \leftarrow x$ ;  
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- 3 Algorithms can be described by finite texts (*algorithms*).

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Key idea

Model algorithms as sets of sequences over appropriate universes of objects.

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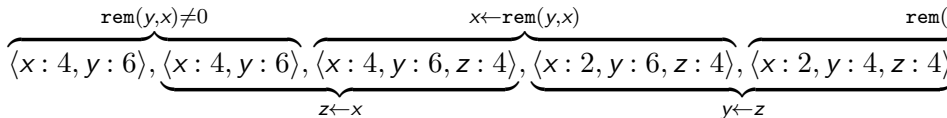
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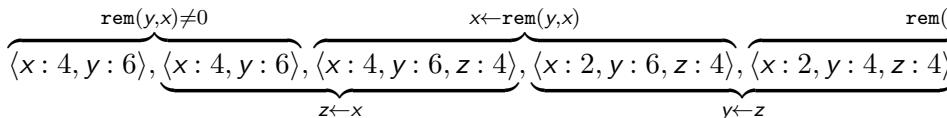
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Question

What does it mean for an algorithm to be constructed from basic operations?

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- If σ and τ haven't yet been distinguished, then the same set of operations is applied to each.

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Define a *control equivalence* relation on the trace points of an algorithm:

$$\sigma \stackrel{c}{\sim} \tau \quad \text{iff} \quad \text{“the same thing happens next” in } \sigma \text{ and } \tau$$

A control equivalence describes how an algorithm is constructed from operations given by a purpose in context.

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Thesis

An algorithm is a set of execution traces equipped with a finite control equivalence.

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Two perspectives on algorithms (at least)

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An orthodox definition

An algorithm is a finite sequence of instructions that:

- is unambiguous;
- is deterministic;
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An algorithm is a non-syntactic object that:

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- Examples abound of algorithms which violate the standard conditions.

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Desideratum An explanation of when executive finiteness *does* hold.

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- 1 A clear distinction between the two perspectives on algorithms:
- 2 A careful analysis of the concept of an algorithm from the distinct methods perspective:
- 3 A formal definition of finite control trace sets for a restricted class of algorithms, with associated results:

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 - The recovery of computability theory using finite control trace sets

Thank you

