Local Fact Change Logic, Memory Logic and Expressive Power

Declan Thompson
Stanford University

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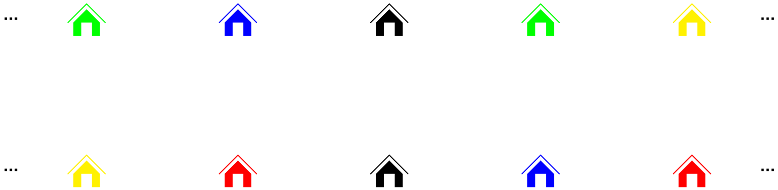
Talk Overview

1. Introduction
2. Local Fact Change
3. Expressive Power
4. Open Problems

A simple game



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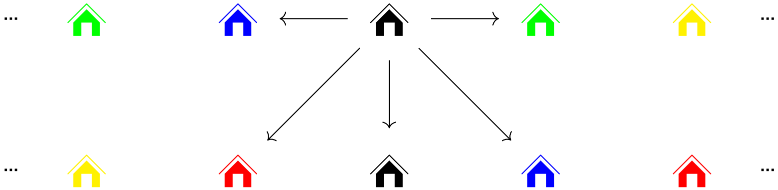

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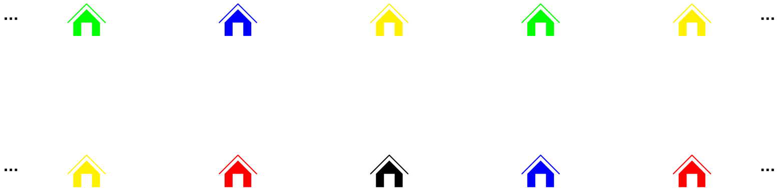

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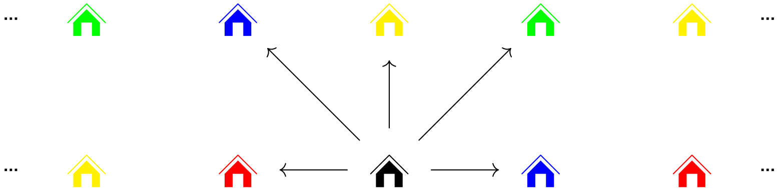

Boolean Network Games

Game Definition

- A set of *players* W
- An accessibility relation $R \subseteq W \times W$
- A goal formula γ

Strategies

A strategy for $s \in W$ is a choice of propositional letters. A strategy profile is a function $V : W \to 2^{\text{Prop}}$, i.e. a valuation on (W, R).

Outcomes

s wins under V iff $(W, R), V, s \models \gamma$.

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A logic for propositional control in a network.

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$\mathfrak{F}, V, s \models \neg\varphi$	iff	$\mathfrak{F}, V, s \not\models \varphi$
$\mathfrak{F}, V, s \models (\varphi \wedge \psi)$	iff	$\mathfrak{F}, V, s \models \varphi$ and $\mathfrak{F}, V, s \models \psi$
$\mathfrak{F}, V, s \models \Diamond\varphi$	iff	$\mathfrak{F}, V, t \models \varphi$ for some t with Rst

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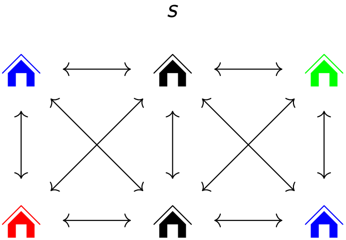
$\mathfrak{F}, V, s \models \bigcirc\varphi$	iff	$\mathfrak{F}, V^s_A, s \models \varphi$ for some $A \subseteq \mathrm{Prop}$

$\bigcirc$ changes the valuation *but only at the current state.*

$\bullet\varphi := \neg\bigcirc\neg\varphi$

Example



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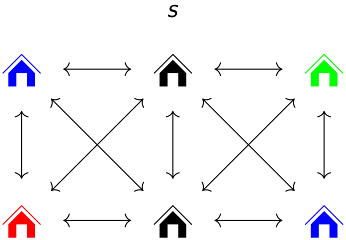

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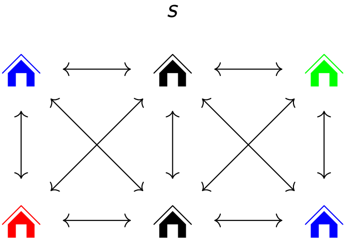

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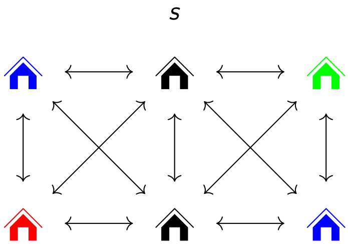

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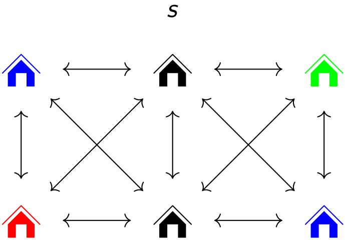

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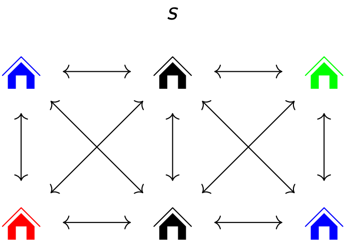

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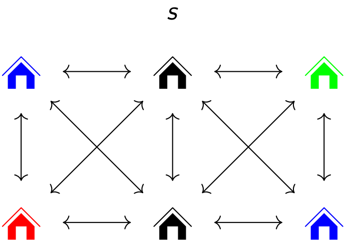

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Nash equilibria

V is a Nash equilibrium iff $\mathfrak{F}, V, t \models \gamma \vee \bullet\neg\gamma$ for all t.

Some initial results about LFC

Fact

$\mathfrak{F}, V, s \models \bigcirc\varphi$ *iff* $\mathfrak{F}, V^s_A, s \models \varphi$ *for some* $A \subseteq \mathrm{At}(\varphi)$.

Translation into FOL

Exists, but with exponential blow-up. Examples:

$$\begin{aligned} \mathsf{T}(\bigcirc p, x, \emptyset) &= Px \vee \neg Px \\ \mathsf{T}(\bigcirc \Diamond p, x, \emptyset) &= \exists y(Rxy \wedge ((x = y \rightarrow \neg Px) \wedge (x \neq y \rightarrow Px))) \\ &\quad \vee \exists y(Rxy \wedge ((x = y \rightarrow Px) \wedge (x \neq y \rightarrow Px))) \end{aligned}$$

Some initial results about LFC

Theorem

Model checking for LFC is PSPACE hard.

Proof.

Via reduction from TQBF. Idea: represent variable x_i by the value of p in state s_i, and give each state a unique label q_i. Then translate x_i to $\Box(q_i \rightarrow p)$. □

- We can show model checking for LFC is in PSPACE by a direct argument, but not via translation to FOL.

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$$\textcircled{p}\varphi := \bigwedge_{\psi \in \mathrm{Bool}(\mathrm{At}(\varphi) \setminus \{p\})} (\psi \to \bigcirc(p \wedge \psi \wedge \varphi))$$

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$$\textcircled{p}(q \vee p) = (q \to \bigcirc(p \wedge q \wedge (q \vee p))) \wedge (\neg q \to \bigcirc(p \wedge \neg q \wedge (q \vee p)))$$

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We say $\mathfrak{F}, V, s \models_{\mathsf{M}} \varphi$ iff $\mathfrak{F}, V, \emptyset, s \models_{\mathsf{M}} \varphi$.

Memory Logic (M)

Examples

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Theorem

The satisfiability problem for memory logic is undecidable (Mera 2009).

Undecidability of LFC

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Main idea

We translate satisfiability problems for M to satisfiability problems for LFC.

$$\tau(p, q) = p \qquad \tau(\varphi \wedge \psi, q) = \tau(\varphi, q) \wedge \tau(\psi, q)$$
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$$\mathsf{T}(\varphi, q) = \tau(\varphi, q) \wedge \bigwedge_{0 \leq i \leq \mathsf{MD}(\varphi)} \Box^i \neg q$$

If $q \notin \mathsf{At}(\varphi)$ then φ is satisfiable in M iff $\mathsf{T}(\varphi, q)$ is satisfiable in LFC.

This translation does not preserve truth.

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This translation does not preserve truth.

T is not truth preserving

$$s \longrightarrow t$$

Suppose $V(t) = \text{Prop}$. Then

$$s \models_{\text{M}} \circledr \Diamond \neg \circledk \qquad\qquad s \not\models_{\text{LFC}} \circled{q} \Diamond \neg q$$

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A question

Does a truth preserving translation exist? What is the relative expressive power of M and LFC?

Comparing expressive power

Definition ($\preceq$, "no more distinctions")

A $\preceq$ B if every pair of models equivalent under B is equivalent under A.

Definition ($\leq$, "translation")

Let A, B be logics. $\mathsf{A} \leq \mathsf{B}$ if there is a translation $\mathfrak{T} : \mathcal{L}_{\mathsf{A}} \to \mathcal{L}_{\mathsf{B}}$ such that for all models $\mathfrak{M}$,

$$\mathfrak{M} \models_{\mathsf{A}} \varphi \quad \text{iff} \quad \mathfrak{M} \models_{\mathsf{B}} \mathfrak{T}(\varphi).$$

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Fact

If A $\leq$ B *then* A $\preceq$ B.

Comparing expressive power

Standard approach

To show A $\leq$ B, provide a translation.
To show A $\not\leq$ B, show A $\not\preceq$ B.

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To show A $\leq$ B, provide a translation.
To show A $\nleq$ B, show A $\not\preceq$ B.

To show M $\nleq$ LFC, it suffices to find a pair of models M can distinguish that LFC cannot.

Ehrenfeucht-Fraïssé Games for LFC

$EF(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, s_1, s_2)$

- *If* $V_1(s_1) \neq V_2(s_2)$ then Spoiler wins.
- *Else* if both s_1 and s_2 have no neighbours then Duplicator wins.
- *Else* Spoiler chooses one of the following two moves:

2. The following occur in order:
   1. Spoiler chooses $i \in \{1, 2\}$ (j is the other)
   2. Spoiler chooses $t_i \in W_i$ such that $R_i s_i t_i$.
   3. If there is no t_j with $R_j s_j t_j$, Spoiler wins. Otherwise, Duplicator picks such a t_j. We play $EF(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, t_1, t_2)$.

In an infinite game, Duplicator wins.

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Fact

If Duplicator has a winning strategy then for all φ, $\mathfrak{F}_1, V_1, s_1 \models_{\mathsf{LFC}} \varphi$ *iff* $\mathfrak{F}_2, V_2, s_2 \models_{\mathsf{LFC}} \varphi$.

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We find a pair of models M can distinguish that LFC cannot.

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$\mathfrak{G}_1$

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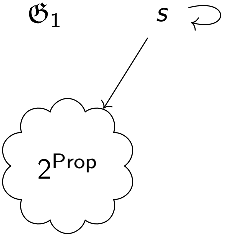

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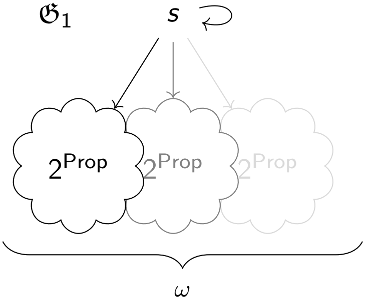
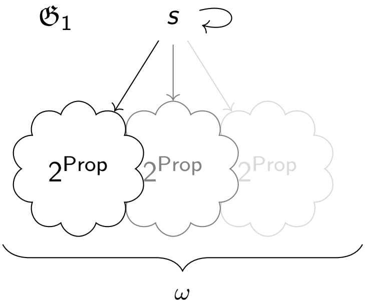


$M \not\leq$ LFC

We find a pair of models M can distinguish that LFC cannot.



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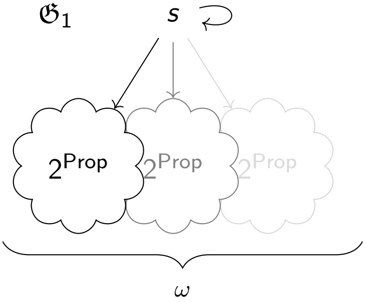
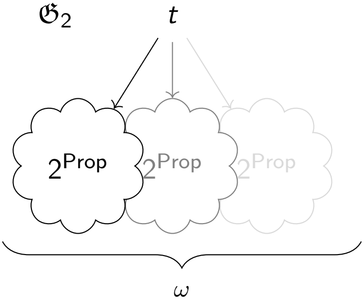
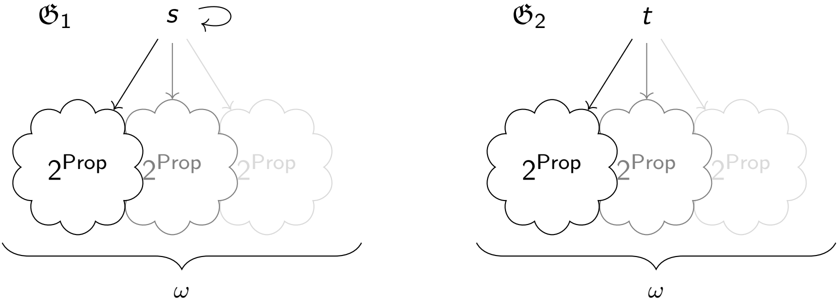




$\mathfrak{G}_1, V, s \models_{\mathsf{M}} ⓡ \Diamond ⓚ$

$\mathfrak{G}_2, V, t \not\models_{\mathsf{M}} ⓡ \Diamond ⓚ$

M $\not\leq$ LFC (cont.)



Duplicator has a winning strategy in LFC

- s and t have the same valuation and every node has a neighbour.
- For every node spoiler picks, there is an *unvisited* node with the same valuation.

M $\nleq$ LFC: Argument summary

M $\nleq$ LFC

M can distinguish $\mathfrak{G}_1$ and $\mathfrak{G}_2$ and LFC cannot

$\Rightarrow$ M $\npreceq$ LFC

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M $\npreceq$ LFC: Argument summary

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Fact

LFC $\npreceq$ M

Proof.

Adaptation of proof in Areces, D. Figueira, S. Figueira, et al. (2011). Uses infinite models to show LFC $\nprec$ M. □

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What about finite models?

Restricted EF Games for LFC

$EF_R(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, s_1, s_2)$

- *If* $V_1(s_1) \neq V_2(s_2)$ then Spoiler wins.
- *Else* if both s_1 and s_2 have no neighbours then Duplicator wins.
- *Else* Spoiler chooses one of the following two moves:
    1. Spoiler picks $A \subseteq \text{Prop}$. We play $EF(\mathfrak{F}_1, \mathfrak{F}_2, {V_1}^{s_1}_A, {V_2}^{s_2}_A, s_1, s_2)$.
    2. The following occur in order:
        1. Spoiler chooses $i \in \{1, 2\}$ (j is the other)
        2. Spoiler chooses $t_i \in W_i$ such that $R_i s_i t_i$.
        3. If there is no t_j with $R_j s_j t_j$, Spoiler wins. Otherwise, Duplicator picks such a t_j. We play $EF(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, t_1, t_2)$.

In an infinite game, Duplicator wins.

Restricted EF Games for LFC

$EF_R(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, s_1, s_2, B \subseteq \text{Prop}$

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Restricted EF Games for LFC

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If Spoiler does not win in n rounds, Duplicator wins.

Fact

If Duplicator has a winning strategy, $\mathrm{MD}(\varphi) \leq n$ *and* $\mathrm{At}(\varphi) \subseteq B$*,*
$\mathfrak{F}_1, V_1, s_1 \models_{\mathsf{LFC}} \varphi$ *iff* $\mathfrak{F}_2, V_2, s_2 \models_{\mathsf{LFC}} \varphi$.

M $\not\leq$ LFC on finite models

Suppose $T : \mathcal{L}_\mathsf{M} \to \mathcal{L}_{LFC}$, and $\psi = T(\textcircled{r} \lozenge \textcircled{k})$.

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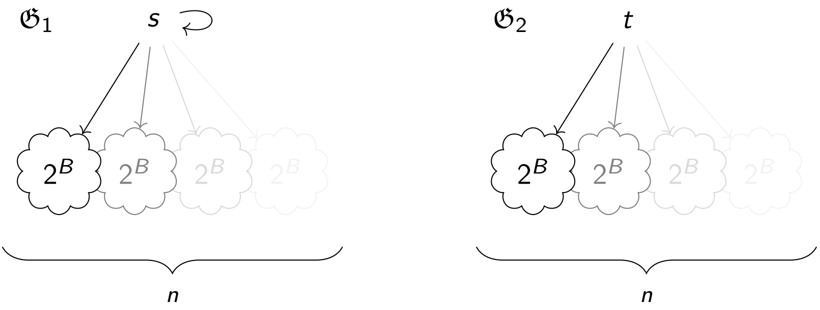

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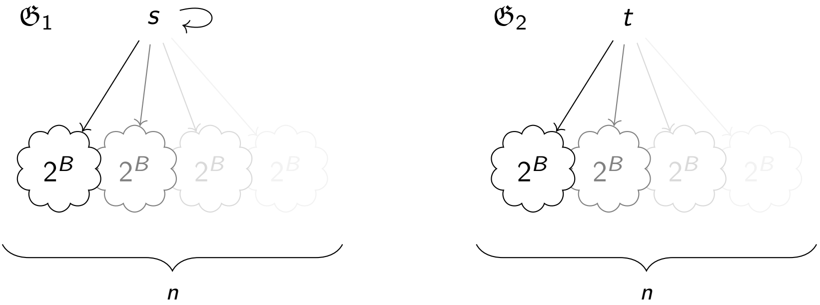

- Duplicator has the same winning strategy as before in $EF_R(\mathfrak{G}_1, \mathfrak{G}_2, V_1, V_2, s, t, B, n)$.

M $\not\leq$ LFC on finite models: Argument summary

M $\not\leq$ LFC on finite models

Suppose $T : \mathcal{L}_{\mathsf{M}} \to \mathcal{L}_{LFC}$.

Construct $\mathfrak{G}_1$ and $\mathfrak{G}_2$

ⓡ $\Diamond$ ⓚ distinguishes $\mathfrak{G}_1$ and $\mathfrak{G}_2$ but $T($ⓡ $\Diamond$ ⓚ$)$ does not

$\Rightarrow$ T is not truth preserving on finite models

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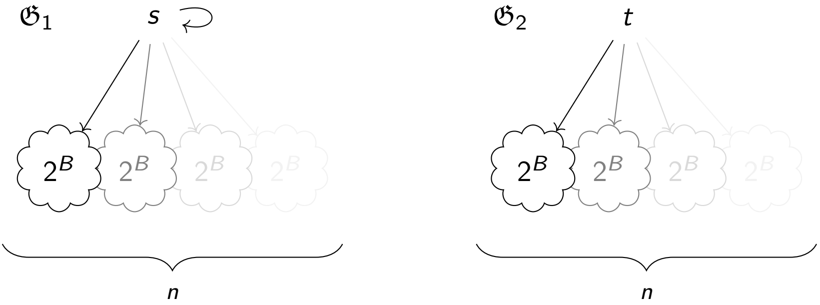
We bypassed $\preceq$.

BUT!

Fix n and $B \subsetneq \mathrm{Prop}$. Take $q \notin B$.



$\mathfrak{G}_1, s \models_{\mathrm{LFC}} \bigcirc \lozenge q$ $\mathfrak{G}_2, t \not\models_{\mathrm{LFC}} \bigcirc \lozenge q$

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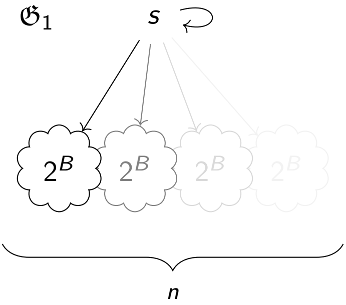

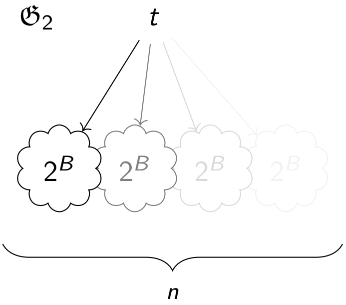

$\mathfrak{G}_1, s \models_{\mathrm{LFC}} \bigcirc \Diamond q$

$\mathfrak{G}_2, t \not\models_{\mathrm{LFC}} \bigcirc \Diamond q$

- So LFC can distinguish all our countermodels.

M $\preceq$ LFC on finite models

Lemma

For finite pointed models $\mathfrak{F}_1, V_1, s_1$ *and* $\mathfrak{F}_2, V_2, s_2$*, the following are equivalent:*

1. *Duplicator has a winning strategy in* $EF(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, s_1, s_2)$*.*
2. *For every* $\varphi \in \mathcal{L}_{LFC}$ *we have* $\mathfrak{F}_1, V_1, s_1 \models_{\mathsf{LFC}} \varphi$ *iff* $\mathfrak{F}_2, V_2, s_2 \models_{\mathsf{LFC}} \varphi$*.*

Theorem

If Duplicator has a winning strategy in $EF(\mathfrak{F}_1, \mathfrak{F}_2, V_1, V_2, s_1, s_2)$*, with* $\mathfrak{F}_1, \mathfrak{F}_2$ *finite, then for all* $\varphi \in \mathcal{L}_{\mathsf{M}}$*,*

$$\mathfrak{F}_1, V_1, s_1 \models_{\mathsf{M}} \varphi \quad \textit{iff} \quad \mathfrak{F}_2, V_2, s_2 \models_{\mathsf{M}} \varphi.$$

Proof.

Via a version of bisimulation for LFC. □

Summary

Corollary

M $\preceq$ LFC *on finite models.*

What do we know?

- M and LFC are incomparable in general.
- M $\not\preceq$ LFC and LFC $\not\preceq$ M for finite models.
- M $\preceq$ LFC for finite models.

Open questions

Expressive Power

- Is LFC $\preceq$ M on finite models?
- What other situations do $\leq$ and $\preceq$ give different judgements?
- What other notions of relative expressive power are interesting?
- Relationship with other logics (e.g. Hybrid logic)?

General

- What is the exact relationship between LFC and Nash equilibria for BNGs? Can we say the logic of Nash equilibria is undecidable?
- What is a natural, efficient translation of LFC to FOL?
- Axiomatisation?
- Tableau system? (c.f. Areces, D. Figueira, Gorín, et al. 2009)

Obtaining decidability

How can we make LFC decidable?

- Restrict the class of models.
  - For DAGs, LFC is decidable.
- Modify $\bigcirc$.
  - Only a subset of valuations are available (c.f. generalised assignment models): remains undecidable.
  - $\bigcirc$ changes the valuation *somewhere* (maybe not here): conjecture remains undecidable.
  - $\bigcirc$ updates all bisimilar points: conjecture becomes decidable.

Thank you

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Van der Hoek, Wiebe and Michael Wooldridge (May 2005). "On the Logic of Cooperation and Propositional Control". In: *Artificial Intelligence* 164.1–2, pp. 81–119.

Related Work

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- PDL with local and global assignments to propositional variables: PDL+GLA (Tiomkin and Makowsky 1985)

Translation into FOL

$$\begin{aligned}
\mathsf{T}(\neg\varphi, x, V) &:= \neg\mathsf{T}(\varphi, x, V)\\
\mathsf{T}(\varphi \vee \psi, x, V) &:= \mathsf{T}(\varphi, x, V) \vee \mathsf{T}(\psi, x, V)\\
\mathsf{T}(\Diamond\varphi, x, V) &:= \exists y(Rxy \wedge \mathsf{T}(\varphi, y, V)) \qquad [y \text{ is } \textit{new}]\\
\mathsf{T}(\bigcirc\varphi, x, V) &:= \bigvee_{A\subseteq \mathsf{At}(\varphi)} \mathsf{T}(\varphi, x, V^x_A)\\
\mathsf{T}(p, x, V) &:= \begin{cases} \neg Px & \text{if } p \in V(y) \text{ for the most recent } y\\ & \text{such that } x = y,\ V(y) \text{ defined}\\ Px & \text{otherwise} \end{cases}
\end{aligned}$$

$$\models \Box^n p \leftrightarrow (\bullet \Box^n p \vee (\bullet(p \leftrightarrow \Box^n p) \wedge p))$$

Definition (Isobisimulation)

Let $\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$. A relation $Z \subseteq W_1 \times W_2$ is an *isobisimulation* if the following clauses hold:

- **Non-empty** $Z \neq \emptyset$
- **Agree** If $s_1 Z s_2$ then $V_1(s_1) = V_2(s_2)$.
- **Zig** If $s_1 Z s_2$ and $R_1 s_1 t_1$ then there is t_2 with $R_2 s_2 t_2$.
- **Zag** If $s_1 Z s_2$ and $R_2 s_2 t_2$ then there is t_1 with $R_1 s_1 t_1$.
- **Isomorphism** If $s_1 Z s_2$ then there is an isomorphism $f : SCC(s_1) \to SCC(s_2)$ such that $f(s_1) = s_2$.

Boolean Games

- We have a set Prop of propositions.
- Each player controls a subset of Prop.
- Each player s has a formula γ_s of propositional logic as their *goal*.
- By choosing the valuation on their propositions, s tries to make γ_s true.

Boolean Network Games

- Players are arranged in a network.
- Each player controls *all the propositions at their position*.
- Each player s has a formula γ_s of *modal* logic as their goal.
- By choosing the valuation *at their position*, s tries to make γ_s true.

BNGs: Strategies and equilibria

Definition (Strategy (profile))

A *strategy* is a subset of Prop. A strategy profile is a function $V : W \to 2^{\text{Prop}}$.

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How can we make this definition more precise? We need a logical way to talk about *changing strategies*.

Equilibria and other properties

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- How expressive is it?