Computable Execution Traces & Algorithms

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Outline

What are algorithms?

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There is no algorithm to decide whether the domain of Φ_x is empty.
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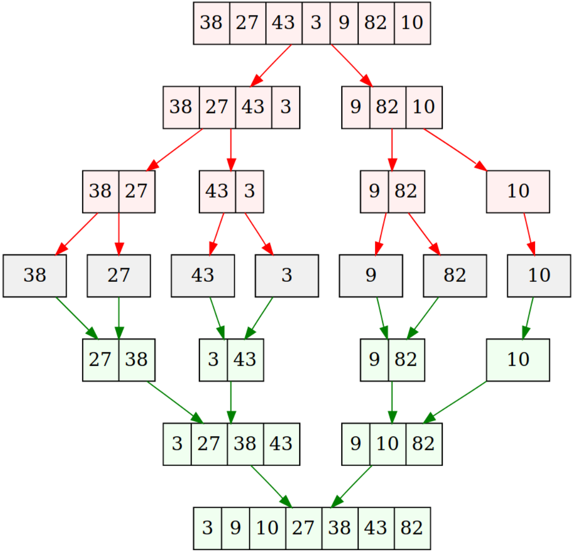
Mergesort sorts a given list of numbers by first dividing them into two equal halves, sorting each half separately by recursion, and then combining the results of these recursive calls–in the form of the two sorted halves–using the linear time algorithm for merging sorted lists that we saw in Chapter 2. [7, p. 210]



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Equivalence class approach

Algorithms as equivalence classes under some equivalence relation. *Dubious that any adequate equivalence relation exists.*

Generalised programs

Allow arbitrary operations and objects in programs. *Language dependent, and over-determined.*

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  4. Ask *what*, not *how*
  5. Show extensional equivalence with other accounts $\rightsquigarrow$ Church-Turing Thesis

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Computable execution traces (cont.)

In all cases, we get a model which takes input and moves through a (possibly infinite) sequences of stages. This gives a set of *execution traces*, built from the bottom up.

$$\textsc{Run}_{\mathfrak{M}} = \left\{ \begin{array}{c} ([|\mathrm{B}], [1|\mathrm{B}], [|1\mathrm{B}], [|\mathrm{BB}]) \\ ([|1], [1|\mathrm{B}], [11|\mathrm{B}], [1|1\mathrm{B}], [|1\mathrm{BB}]) \\ ([|1\mathrm{B}1], [1|\mathrm{B}1], [11|1], [111|\mathrm{B}], [11|1\mathrm{B}], [1|1\mathrm{BB}]) \\ \vdots \end{array} \right\}$$

The focus is on what can be computed, not the way the computation proceeds.

Motivating question

What is required for a given set of sequences to be the execution trace set of some computable process?

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- If $\sigma[-1] = \tau[0]$ then $\sigma \circ \tau = (\!|\sigma[0], \sigma[1], \ldots \sigma[-1], \tau[1], \tau[2], \ldots|\!)$.

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- $\mathrm{F} \circ \mathrm{G} = \{\sigma \circ \tau \mid \sigma \in \mathrm{F},\ \tau \in \mathrm{G}\}$.
  - Q is a *test* if $|\sigma| = 1$ for all $\sigma \in \mathrm{Q}$. $\mathrm{F} \circ \mathrm{Q} = \{\sigma \in \mathrm{F} \mid \sigma[-1] \in \mathrm{Q}\}$
  - P is an *operation* if $|\sigma| = 2$ for all $\sigma \in \mathrm{P}$. $\mathrm{F} \circ \mathrm{P} = \{\sigma \mid \sigma[0, -1) \in \mathrm{F},\ (\!|\sigma[-2], \sigma[-1]|\!) \in \mathrm{P}\}$

Which tasks could be execution trace sets?

Idea: Assemble tasks from primitive operations

- Q is a *test* if $|\sigma| = 1$ for all $\sigma \in \mathrm{Q}$
- P is an *operation* if $|\sigma| = 2$ for all $\sigma \in \mathrm{P}$
- $\mathrm{Q} \circ \mathrm{P}$ is a *guarded operation*
- $\mathrm{F} \circ \mathrm{Q} \circ \mathrm{P} = \{\sigma \mid \sigma[0, -1) \in \mathrm{F},\ \sigma[-2] \in \mathrm{Q},\ (\!|\sigma[-2], \sigma[-1]|\!) \in \mathrm{P}\}$

- Classical accounts utilise a finitary control mechanism.
- This gives an equivalence relation on stages of a computation, similar to a bisimulation:
  1. $\sigma[0, 1) \leftrightarrows \tau[0, 1)$ for all $\sigma, \tau \in \mathrm{F}$
  2. if $\sigma[0, \alpha + 1) \leftrightarrows \tau[0, \alpha + 1)$ then "the same thing" happens in each

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$$
\begin{aligned}
\mathrm{TT}_{\mathsf{a}} &:= \{ (\![\mathtt{u} | \mathtt{av}]\!) \mid \mathtt{u}, \mathtt{v} \in \Gamma^* \} \\
\mathrm{TM}_{\mathsf{a},\mathsf{R}} &:= \{ (\![\mathtt{ub} | \mathtt{cv}], [\mathtt{uba} | \mathtt{v}]\!) \mid \mathtt{u}, \mathtt{v} \in \Gamma^* \ \mathsf{b}, \mathsf{c} \in \Gamma \} \\
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\mathfrak{q} &= \{ \mathrm{TT}_{\mathsf{a}} \mid \mathsf{a} \in \Gamma \} \\
\mathfrak{p} &= \{ \mathrm{TM}_{\mathsf{a},\mathsf{R}}, \mathrm{TM}_{\mathsf{a},\mathsf{L}} \mid \mathsf{a} \in \Gamma \}
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Set $\sigma \leftrightarrows \tau$ iff they correspond to the same internal state of $\mathfrak{M}$.

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$$(\![[|\mathtt{B}]]\!) \leftrightarrows (\![[|\mathtt{1}]]\!) \leftrightarrows (\![[|\mathtt{1B1}]]\!) \leftrightarrows \ldots$$

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- $\sigma[0,1) \leftrightarrows \tau[0,1)$ for all $\sigma, \tau \in \mathrm{F}$.
- If $\sigma \leftrightarrows \tau$ then "the same thing" should happen at each.
  - How to capture this?

Control Equivalence (cont.)

Take an equivalence class C under $\leftrightarrows$.

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- There is a *construction set* of guarded operations $\mathfrak{F}_C \subseteq \mathfrak{q} \times \mathfrak{p}$ such that
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  2. If a guarded operation is successfully applied to σ and τ, the results are equivalent under $\leftrightarrows$.

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$$\mathfrak{F}_C = \begin{cases} \{\langle \mathrm{TT}_1, \mathrm{TM}_{1,\mathsf{R}} \rangle, \langle \mathrm{TT}_\mathtt{B}, \mathrm{TM}_{1,\mathsf{R}} \rangle\} & s = s_0 \\ \{\langle \mathrm{TT}_1, \mathrm{TM}_{1,\mathsf{R}} \rangle, \langle \mathrm{TT}_\mathtt{B}, \mathrm{TM}_{\mathtt{B},\mathsf{L}} \rangle\} & s = s_1 \\ \{\langle \mathrm{TT}_1, \mathrm{TM}_{\mathtt{B},\mathsf{L}} \rangle\} & s = s_2 \\ \emptyset & s = s_3. \end{cases}$$

$$(\![\,[|1\mathtt{B}11], [1|\mathtt{B}11], [11|11]\,]\!) \leftrightarrows (\![\,[|1\mathtt{B}11], [1|\mathtt{B}11], [11|11], [111|1], [1111|\mathtt{B}]\,]\!) \qquad (s_1)$$

Control Equivalence (cont.)

Take an equivalence class C under $\leftrightarrows$.

- There is a *construction set* of guarded operations $\mathfrak{F}_C \subseteq \mathfrak{q} \times \mathfrak{p}$ such that
  1. Every guarded operation in $\mathfrak{F}_C$ gets applied to every $\sigma \in C$; *and*
  2. If a guarded operation is successfully applied to σ and τ, the results are equivalent under $\leftrightarrows$.

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$$\mathfrak{G}_C = \begin{cases} \emptyset & s = s_0 \\ \emptyset & s = s_1 \\ \{\mathrm{TT_B}\} & s = s_2 \\ \{\mathrm{TT_1}, \mathrm{TT_B}\} & s = s_3. \end{cases}$$

$$(\!|[|1\mathrm{B}11], [1|\mathrm{B}11], [11|11], [111|1], [1111|\mathrm{B}], [111|1\mathrm{B}], [11|1\mathrm{BB}]|\!) \qquad (s_3)$$

Control Equivalence

Definition (Control Equivalence)

Let $\mathfrak{q}$ be a set of tests and $\mathfrak{p}$ of operations. Let F be a task. A *control equivalence* $\leftrightarrows$ on F under $\mathfrak{q}$ and $\mathfrak{p}$ is an equivalence relation on $\boxed{\mathrm{F}}$ satisfying:

Starting State $\sigma[0,1) \leftrightarrows \tau[0,1)$ for all $\sigma, \tau \in \mathrm{F}$.

Construction For each $C \in \boxed{\mathrm{F}}/\leftrightarrows$ there is a *construction set* $\mathfrak{F}_C \subseteq \mathfrak{q} \times \mathfrak{p}$ such that both:

- **Composition** $\{\sigma \in \boxed{\mathrm{F}} \mid \sigma[0,-1) \in C\} = \bigcup_{\langle \mathrm{Q},\mathrm{P}\rangle \in \mathfrak{F}_C} C \circ \mathrm{Q} \circ \mathrm{P}$; *and*
- **Consistency** if $\langle \mathrm{Q}, \mathrm{P}\rangle \in \mathfrak{F}_C$ then $C \circ \mathrm{Q} \circ \mathrm{P} \subseteq D$ for some $D \in \boxed{\mathrm{F}}/\leftrightarrows$.

Halting For each $C \in \boxed{\mathrm{F}}/\leftrightarrows$ there is a *halting set* $\mathfrak{G}_C \subseteq \mathfrak{q}$ such that $C \cap \mathrm{F} = \bigcup_{\mathrm{Q} \in \mathfrak{G}_C} C \circ \mathrm{Q}$.

Definition (Trace set)

A *trace set* for $\mathfrak{q}$ and $\mathfrak{p}$ is a pair $\mathrm{A} = (\mathrm{F}, \leftrightarrows)$, where F is a task and $\leftrightarrows$ is a control equivalence on F under $\mathfrak{q}$ and $\mathfrak{p}$.

Finite Control Computability

Proposition

For every task F *over* $\mathfrak{D}$ *there is a test set* $\mathfrak{e}$*, an operation set* $\mathfrak{w}$ *and a control equivalence* $\leftrightarrows$ *such that* $\mathrm{A} = (\mathrm{F}, \leftrightarrows)$ *is a trace set for* $\mathfrak{e}$ *and* $\mathfrak{w}$*.*

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A trace set A for $\mathfrak{q}$ and $\mathfrak{p}$ is *finite control computable* if $\boxed{\mathrm{A}}/{\leftrightarrows_{\mathrm{A}}}$ is finite, $\mathfrak{q}$ and $\mathfrak{p}$ are finite, and every $\mathrm{Q} \in \mathfrak{q}$ and $\mathrm{P} \in \mathfrak{p}$ is computable.

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Theorem

If $\mathfrak{M} = \langle S, \Gamma, \delta, s_0 \rangle$ *is a Turing machine then* $\textsc{Run}_{\mathfrak{M}}$ *is finite control computable.*

Does finite control computable imply Turing computable?

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Is every FCC trace set the trace set of some Turing machine?

No!

1. FCC allows arbitrary computable tests and operations
2. FCC allows more domains than machine tapes

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Let $\mathrm{A} = (\mathrm{F}, \leftrightarrows)$ *be a trace set for* $\mathfrak{t}$ *and* $\mathfrak{m}$ *and* $\Gamma \subseteq \mathbb{G}$ *a finite alphabet such that*

1. *For every* $\sigma \in \mathrm{A}$ *and* $\alpha < |\sigma|$, $\sigma[\alpha] = [\mathrm{u}|\mathrm{v}]$ *for some* $\mathrm{u}, \mathrm{v} \in \Gamma^*$*; and*
2. *For every* $\mathrm{u}, \mathrm{v} \in \Gamma^*$ *there is a unique* $\sigma \in \mathrm{A}$ *with* $\sigma[0] = [\mathrm{u}|\mathrm{v}]$*; and*
3. A *is fully deterministic (*$\sigma[0] = \tau[0]$ *implies* $\sigma = \tau$*).*

Then there is a Turing machine $\mathfrak{M} = \langle S, \Gamma, \delta, s_0 \rangle$ *such that* $\mathrm{RUN}_{\mathfrak{M}} = \mathrm{A}$ *iff* A *is finite control computable.*

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Can we do better? Can we allow arbitrary computable tests and operations?

Expansion mapping

$$\llparenthesis [|\mathtt{1B11}], [\mathtt{11}|\mathtt{11}], [\mathtt{111}|\mathtt{1B}], [\mathtt{11}|\mathtt{1BB}] \rrparenthesis$$

Expansion mapping

$$(\!([\,|\texttt{1B11}],\,[\texttt{11}|\texttt{11}],[\texttt{111}|\texttt{1B}],[\texttt{11}|\texttt{1BB}])\!)$$

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No Turing Machine can do this. But $\mathfrak{M}$ can fill in the gaps:

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Define a function h mapping prefixes of the original sequence to prefixes of the expanded sequence.

$$h((\![\,|1B11]\!)) = (\![\,|1B11]\!)$$
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Expansion Mapping

$\mathrm{F} \ll \mathrm{G}$ if there is an injective $h : \boxed{\mathrm{F}} \to \boxed{\mathrm{G}}$ such that σ is a subsequence of $h(\sigma)$ and $h(\mathrm{F}) = \mathrm{G}$.

Expansion mapping is not enough

Example

Take $\textsc{Primes}$, enumerating all prime in increasing order:

$$\{ (\!| [\mathtt{11}|\mathtt{B}], [\mathtt{111}|\mathtt{B}], [\mathtt{11111}|\mathtt{B}], [\mathtt{1111111}|\mathtt{B}], \ldots |\!) \}$$

Take $\mathfrak{M} = \langle \{s\}, \{1, B\}, \delta, s \rangle$ with $\delta(s, 1) = \delta(s, \mathtt{B}) = \langle 1, \mathsf{R}, s \rangle$.

$$\{ (\!| [|\mathtt{B}], [\mathtt{1}|\mathtt{B}], [\mathtt{11}|\mathtt{B}], [\mathtt{111}|\mathtt{B}], [\mathtt{1111}|\mathtt{B}], \ldots |\!) \}$$

Carrying out

Two requirements:

1. We need to be able to recover the original task from the expanded task.
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Definition (Carrying out)

We say A is *carried out* by B if there is an expansion mapping h from A to B such that $C \in \boxed{\mathrm{A}}/\leftrightarrows_{\mathrm{A}}$ iff $h(C) \in \boxed{\mathrm{B}}/\leftrightarrows_{\mathrm{B}}$.

Theorem

Let $\mathrm{A} = (\mathrm{F}, \leftrightarrows)$ *be a trace set for* $\mathfrak{q}$ *and* $\mathfrak{p}$ *such that* A *is finite control and fully deterministic;* $\mathfrak{q}$ *and* $\mathfrak{p}$ *are both finite and Turing computable; there is a finite alphabet* Γ *such that*

1. *for every* $\sigma \in \mathrm{A}$, $\sigma[0] = [\mathrm{u}|\mathrm{v}]$ *for some* $\mathrm{u} \in \Gamma^*, \mathrm{v} \in \Gamma^+$*;*
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FCC and Turing Computability

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Theorem

Let A *be a trace set and* $\mathfrak{M} = \langle S, \Gamma, \delta, s_0 \rangle$ *be a Turing machine. If* $\mathrm{RUN}_{\mathfrak{M}}$ *carries out* A *then* A *is finite control computable.*

Outline

1. What are algorithms?
2. Computable Execution Traces
   - Finite control computability
   - Carrying out trace sets
3. Algorithms as trace sets

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Idea: Model algorithms as trace sets.

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- What model/what objects?

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- Allow arbitrary domains and operations

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Algorithms as trace sets: further details

- Algorithms are always presented as a (potential) solution to some problem, in a particular context.
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  - Finite control ensures this for trace sets.
- Algorithms are ambiguous!

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1. Ask me for details about:
   - Encoding: changing the domain over which a trace set is defined.
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   - Implementation: increasing the level of specification; the relationship between programs and algorithms.
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   - Computable functions vs. algorithms.
2. Ask me to ramble about:
   - Trace sets with arbitrary tasks (not just tests and operations)
   - Recursive algorithms
   - Concurrent/parallel computation
   - Interactive algorithms
   - Transfinite sequences
   - Probabilistic algorithms
   - Complexity theory

Thank you!

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